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Infinitesimal Holomorphic Variations of Invariant Submanifolds of Almost Kaehlerian Manifolds

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Abstract: Goldstein and Ryan (1975) have calculated Infinitesimal rigidity of submanifolds. In this paper, we have defined and studied infinitesimal holomorphic variations of invariant submanifolds of almost Kaehlerian manifolds and some theorems established.

Keywords: kaehlerian manifolds • invariant submanifolds • infinitesimal variations • hermitian metric tensor • riemannian manifolds.

MSC 2020: 32C15, 46A13, 53B35, 53C55, 53B20, 53B30.

1. Introduction

Consider a real $2m$-dimensional Kaehlerian manifold denoted like M^{2m} is prepared with a set of coordinate neighborhoods $\{U; x^h\}$, in using indices $h, i, j,\ etc.,$ which range $\{1,2,..,2m\}$ and n-dimensional Riemannian manifold (M^n), prepared with a set of coordinate neighborhoods $\{V; y^\alpha\}$, in the indices $a, b, c, ...$etc, at range $\{1,2,...,n\}$ an almost complex structure tensor F_i^h and a Hermitian metric tensor g_{ji}. Then, we have obtained:

$$F_i^t F_t^h = -\delta_i^h, F_j^t F_i^s g_{ts} = g_{ji}, \tag{1.1}$$

$$\nabla_j F_i^h = 0, \tag{1.2}$$

∇_j represents the operator for covariant differentiation, which operates in relation to the Christoffel symbols Γ_j^h, constructed using the metric tensor g_{ji}.

If M^n is isometrically fixed within M^{2m} during the immersion map $i: M^n \to M^{2m}$, and we recognize $i(M^n)$ by M^n itself. Also, we describe this immersion as $x^h = x^h(y^\alpha)$ and define $B_b^h = \partial_b x^h$, $\partial_b = \partial/\partial y^b$, which are n linearly independent lie tangent to M^n within the larger manifold M^{2m}. Since the immersion i preserves distances, we can express this as follows:

$$g_{cb} = g_{ji} B_c^j B_b^i, \tag{1.3}$$

Again, if C_y^h is $(2m-n)$equally orthogonal unit normal to M^n in the indices $x, y, z, \dots$path ended the range $\{n+1, n+2, \dots, 2m\}$. Hence Guass equation is:

$$\nabla_c B_b^h = h_{cb}^x C_x^h, \tag{1.4}$$

Here, ∇_c represents the symbol of covariant differentiation along M^n and using Γ_{ji}^h derived form g_{ji}, along with Γ_{cb}^a formed using g_{cb}. Additionally, we have the second fundamental tensors h_{cb}^x of M^n concerning the normal vector C_x^h, and Weingarten tensors.

$$\nabla_c C_x^h = - h_{cx}^a B_a^h, \tag{1.5}$$

Also, if g_{zx} is metric tensor of the regular bundle, then we get:

$$h_{cx}^a = h_{cbx} g^{ba} = h_{cb}^z g^{ba} g_{zx}, \left(g^{ba}\right) = (g_{ba})^{-1},$$

If F change any vector lie tangent to M^n, then results in another vector that remains tangent to M^n, it implies the existence of a type $(1,1)$ tensor field denoted as f_b^a on M^n. That is M^n is invariant in M^{2m} , When it comes to transformations performed by F or the normal C_y^h , we come upon equations that are structured like so:

$$F_i^h B_b^i = f_b^a B_a^h, \quad F_{ih} B_b^i C_x^h = 0, \text{ where } F_{ih} = F_i^t g_{th}. \tag{1.6}$$

$$F_i^h C_y^i = f_y^x C_x^h. \tag{1.7}$$

We put $f_{yx} = f_y^z g_{zx}$, then we have $f_{yx} = - f_{xy}$.
From (1.1), (1.3), (1.6) and (1.7), we easily see that:

$$f_b^e f_e^a = - \delta_b^a, \quad f_c^e f_b^d g_{ed} = g_{cb}, \tag{1.8}$$

$$f_y^z f_z^x = - \delta_y^x. \tag{1.9}$$

Differentiating (1.6) and (1.7) covariantly along M^n and using (1.2), (1.4) and (1.5), we find:

$$\nabla_c f_b^a = 0, \tag{1.10}$$

$$\nabla_c f_y^x = 0, \tag{1.11}$$

$$h_{cb}^y f_y^x = h_{ce}^x f_b^e \tag{1.12}$$

Thus, equations (1.8) and (1.10) show that M^n is also Kaehlerian manifolds. Moreover, it follows from (1.12), that is, M^n is minimal, then we get:

$$h_{ey}^e = 0, \tag{1.13}$$

Using (1.8), (1.9) and (1.12) we easily verify that:

$$h_{cb}^x = - h_{ed}^x f_c^e f_b^d. \tag{1.14}$$

The Equations of Gauss and Codazzi for the submanifold M^n are expressed as follows:

$$K_{dcb}^{a} = K_{kji}^{h} B_{d}^{k} B_{c}^{j} B_{b}^{i} B_{h}^{a} + h_{dx}^{a} h_{cb}^{x} - h_{cx}^{a} h_{db}^{x}, \tag{1.15}$$

$$K_{kji}^{h} B_{d}^{k} B_{c}^{j} B_{b}^{i} C_{h}^{x} - \left(\nabla_{d} h_{cb}^{x} - \nabla_{c} h_{db}^{x}\right) = 0, \tag{1.16}$$

Where K_{dcb}^{a} is the curvature tensor of M^{n}.

Lastly, we establish a useful identity on a Kaehlerian manifold M^{n}, that is:

$$\frac{1}{2} f^{ce} f_{b}^{d} K_{ceda} = K_{ab}. \tag{1.17}$$

2. Infinitesimal Variation of Invariant Submanifolds.

We examine an infinitesimal variation of the invariant submanifold M^{n}within the context of the Kaehlerian manifold M^{2m}, thus

$$\bar{x}^{h} = x^{h}(y) + \xi^{h}(y)\varepsilon, \tag{2.1}$$

Here $\xi^{h}(y)$ represents a vector field on M^{2m} defined along M^{n} and ε is an infinitesimal. Then we get:

$$\bar{B}_{b}^{h} = B_{b}^{h} + \left(\partial_{b} \xi^{h}\right)\varepsilon, \tag{2.2}$$

Here, we have $\bar{B}_{b}^{h} = \partial_{b} \bar{x}^{h}$, which represents a set of linearly independent tangent vectors on the perturbed submanifold. We proceed to displace these $\bar{B}_{b}^{h}$vectors in a parallel, transitioning them from their positions at the perturbed point $(\bar{x}^{h})$ to the reference point (x^{h}). This displacement leads to the derivation of new vectors. We get:

$$\tilde{B}_{b}^{h} = \bar{B}_{b}^{h} + \Gamma_{ji}^{h}(x + \xi\varepsilon)\xi^{j} \bar{B}_{b\varepsilon}^{i}$$

And at the point (x^{h}), then:

$$\tilde{B}_{b}^{h} = B_{b}^{h} + \left(\nabla_{b} \xi^{h}\right)\varepsilon, \tag{2.3}$$

Disregarding higher order terms with respect to ε, we get:

$$\nabla_{b} \xi^{h} = \partial_{b} \xi^{h} + \Gamma_{ji}^{h} B_{b}^{j} \xi^{i}. \tag{2.4}$$

Throughout the discussion, we consistently disregard terms beyond first order concerning ε. As a result, we can express this as follows:

$$\delta B_{b}^{h} = \tilde{B}_{b}^{h} - B_{b}^{h}, \tag{2.5}$$

We have from (2.3)

$$\delta B_{b}^{h} = \left(\nabla_{b} \xi^{h}\right)\varepsilon. \tag{2..6}$$

Putting:

$$\xi^{h} = \xi^{a} B_{a}^{h} + \xi^{x} C_{x}^{h}, \tag{2.7}$$

We have

$$\nabla_{b} \xi^{h} = \left(\nabla_{b} \xi^{a} - h_{bx}^{a} \xi^{x}\right) B_{a}^{h} + \left(\nabla_{b} \xi^{x} + h_{ba}^{x} \xi^{a}\right) C_{x}^{h}. \tag{2.8}$$

Because of (1.4) and (1.5).

Now, let $\widetilde{C}_y^h$ represent $(2m-n)$ mutually orthogonal unit normals to the varied submanifold, obtained by parallel displacement from the point $(\tilde{x}^h)$ to (x^h). In this context, we obtain:

$$\widetilde{C}_y^h = \overline{C}_y^h + \Gamma_{ji}^h(x+\xi\varepsilon)\xi^j\overline{C}_y^i\varepsilon. \quad (2.9)$$

We put

$$\delta C_y^h = \widetilde{C}_y^h - C_y^h. \quad (2.10)$$

And assume that δC_y^h is of the form

$$\delta C_y^h = \eta_y^h\varepsilon = \left(\eta_y^a B_a^h + \eta_y^x C_x^h\right)\varepsilon. \quad (2.11)$$

Then, form (2.9), (2.10) and (2.11), we have

$$\overline{C}_y^h = C_y^h - \Gamma_{ji}^h\xi^j C_y^i\varepsilon + \left(\eta_y^a B_a^h + \eta_y^x C_x^h\right)\varepsilon. \quad (2.12)$$

Applying the operator δ to $B_b^j C_y^i g_{ji} = 0$ and using (2.6), (2.8), (2.11) and $\delta g_{ji} = 0$, we find

$$\left(\nabla_b\xi_y + h_{bay}\xi^a\right) + \eta_{yb} = 0,$$

Where $\xi_y = \xi^z g_{zy}$ and $\eta_{yb} = \eta_y^c g_{cb}$, or

$$\eta_y^a = -\left(\nabla^a\xi_y + h_{by}^a\xi^b\right), \quad (2.13)$$

Where $\nabla^a = g^{ac}\nabla_c$, and δ is $C_y^j C_x^i g_{ji} = g_{yx}$, and then (2.11) and $\delta g_{ji} = 0$, we find

$$\eta_{yx} + \eta_{xy} = 0, \quad (2.14)$$

Where $\eta_{yx} = \eta_y^z g_{zx}$.

We assume that the infinitesimal change given by (2.1) remains constant on an invariant submanifold, then

$F_i^h(x+\xi\varepsilon)\overline{B}_b^i$ are linear combinations of $\overline{B}_b^h$. (2.15)

Then, using $\nabla_j F_i^h = 0$ and (1.6), we see that

$$F_i^h(x+\xi\varepsilon)\overline{B}_b^i = \left(F_i^h + \xi^j\partial_j F_i^h\varepsilon\right)\left(B_b^i + \partial_b\xi^i\varepsilon\right)$$

$$= \left[F_i^h - \xi^j\left(\Gamma_{jt}^h F_i^t - \Gamma_{ji}^t F_t^h\right)\varepsilon\right]\left(B_b^i + \partial_b\xi^i\varepsilon\right)$$

$$= F_i{}^h B_b{}^i + \left(F_i{}^h\nabla{}_b\xi^i - f_b^a\Gamma_j{}^h{}_i B_a{}^j\xi^i\right)\varepsilon ,$$

That is, by (2.2)

$$F_i^h(x+\xi\varepsilon)\overline{B}_b^i = f_b^a\overline{B}_a{}^h + \left[F_i{}^h\nabla_b\xi^i - f_b^a\nabla_a\xi^h\right]\varepsilon , \quad (2.16)$$

Or, using (2.8),

$$F_i{}^h(x+\xi\varepsilon)\overline{B}_b^i = f_b^a\overline{B}_a{}^h + f_e^a\left(\nabla_b\xi^e + h_{bx}^e\xi^x\right)\overline{B}_a{}^h\varepsilon \quad (2.17)$$

$$+\left(\nabla{}_b\xi^y + h_{ba}{}^y\xi^a\right)f_y{}^x\overline{C}_x{}^h\varepsilon - f_b{}^a\left(\nabla{}_a\xi^e - h_a{}^e{}_x\xi^x\right)\overline{B}_e{}^h\varepsilon - f_b{}^e\left(\nabla{}_e\xi^x + h_{ec}{}^x\xi^c\right)\overline{C}_x{}^h\varepsilon .$$

Thus (2.15) is equivalent to

$$\left(\nabla_b\xi^y + h_{bc}^y\xi^c\right)f_y^x = f_b^e\left(\nabla_e\xi^x + h_{ce}^x\xi^c\right) , \quad (2.18)$$

Or, by (1.12), to

$$\left(\nabla_b\xi^y\right)f_y^x = f_b^e\left(\nabla_e\xi^x\right). \quad (2.19)$$

Now, we can state the following theorems:

Theorem 2.1 To establish an infinitesimal variation as complex, then both Necessary and sufficient for the variation vector ξ^h satisfy equation (2.19).

Theorem 2.2 If a vector field ξ^h induces a complex variation, then a Vector field ξ'^h with an identical normal component as ξ^h possesses the same characteristic.

Proof: Consider an infinitesimal variation described by equation (2.1), which changes a submanifold d$x^h = x^h(y)$ into another submanifold $\bar{x}^h = \bar{x}^h(y)$, while maintaining the parallelism of the tangent space of the original submanifold at (x^h) and the perturbed submanifold at the corresponding point $(\bar{x}^h)$. In this case, we pass on to this perturbation asa parallel variation, as deduced from equations (2.5), (2.6), and (2.8).

$$\tilde{B}_b^h = \left[\delta_b^a + \left(\nabla_b \xi^a - h_{bx}^a \xi^x\right)\varepsilon\right] B_a^h + \left(\nabla_b \xi^x + h_{ba}^x \xi^a\right) C_x^h \varepsilon, \tag{2.20}$$

Here, an infinitesimal variation to be parallel, it is necessary and sufficient condition that

$$\nabla_b \xi^x + h_{ba}^x \xi^a = 0. \tag{2.21}$$

If condition (2.21) holds, then it implies the satisfaction of condition (2.19),
Consequently, we can conclude:

Theorem 2.3 A parallel variation necessarily qualifies as a complex variation.

3. Infinitesimal Holomorphic Variations of Invariant Submanifolds of almost Kaehlerian manifold:

Suppose that an infinitesimal variation $\bar{x}^h = x^h + \xi^h \varepsilon$ carries an invariant submanifold transforming it, implies undergoes a complex variation. We get:

$$F_i{}^h(x + \xi\varepsilon)\bar{B}_b^i - (f_b^a + \delta f_b^a)\bar{B}_a^h, \tag{3.1}$$

We have from (5.17) and (5.18)

$$\delta f_b^a = \left[\left(\nabla_b \xi^e - h_{bx}^e \xi^x\right) f_e^a - f_b^e\left(\nabla_e \xi^a - h_{ex}^a \xi^x\right)\right]\varepsilon. \tag{3.2}$$

From this fact we conclude following points:

(i) Assuming to an infinitesimal variation is complex. Therefore, express the variation of f_b^a using equation (3.2).

We establish the definition of T_{cb} as follows:

$$T_{cb} = \nabla_c \xi_b - f_c^e f_b^d \nabla_e \xi_d - 2h_{cbx}\xi^x. \tag{3.3}$$

Equations (3.2) and (3.3) imply that $\delta f_b^a = 0$ is equivalent to $T_{cb} = 0$ because of (2.8) and (2.14).In the presence of a complex variation that maintains the integrity of f_b^a, we label it as holomorphic. According to equations (3.2), (3.3) as well as the remark provided earlier, then

(ii) A complex variation is deemed holomorphic iff observe of $\nabla_b \xi^a - h_{bx}^a \xi^x$ with f_b^a, that is, $\left(\nabla_b \xi^e - h_{bx}^e \xi^x\right) f_e^a - f_b^e\left(\nabla_e \xi^a - h_{ex}^a \xi^x\right) = 0$ or equivalently $T_{cb} = 0$.

Hence, relating the operator δ to (4.3) with using (5.6), (5.8), $\delta g_{ji} = 0,$ Then:

$$\delta g_{cb} = \left(\nabla_b \xi_c - \nabla_c \xi_b - 2h_{cbx}\xi^x\right)\varepsilon, \tag{3.4}$$

From which:

$$\delta g^{ba} = -\left(\nabla^b \xi^a + \nabla^a \xi^b - 2h_x^{ba}\xi^x\right)\varepsilon. \tag{3.5}$$

A variation applied to a submanifold that results in $\delta g_{cb} = 0$ is termed "isometric" whereas when δg_{cb} is proportional to g_{cb}, it is referred to as "conformal". Consequently, we can conclude that:

(iii) To be consider eudiometric or conformal, a variation of a submanifold must meet both necessary and sufficient conditions such that:

$$\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cbx}\xi^x = 0, \tag{3.6}$$

Or

$$\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cbx}\xi^x = 2\lambda g_{cb}, \tag{3.7}$$

Respectively, λ is a specific function that is defined as:

$$\lambda = \frac{1}{n}\left(\nabla_c \xi^c - h^e_{ex}\xi^x\right). \tag{3.8}$$

We now put:

$$\bar{\Gamma}^a_{cb} = \left(\partial_c \bar{B}^h_b + \Gamma^h_{ji}(\bar{x})\bar{B}^j_c \bar{B}^i_b\right)\bar{B}^a_h \tag{3.9}$$

And

$$\delta\Gamma^a_{cb} = \bar{\Gamma}^a_{cb} - \Gamma^a_{cb},$$

Here, $\bar{\Gamma}^a_{cb}$ represents the Christoffel symbols associated with the warped submanifold.

From (2.2) and (2.20) and (3.9), we obtain:

$$\delta\Gamma^a_{cb} = \left[\left(\nabla_c \nabla_b \xi^h + K^h_{kji}\xi^k B^j_c B^i_b\right)B^a_h + h^x_{cb}\left(\nabla^a \xi_x + h^a_{dx}\xi^d\right)\right]\varepsilon, \tag{3.10}$$

From this, utilizing Gauss's equations (2.15) and Codazzi's equations (2.16) for the submanifolds, we obtain:

$$\delta\Gamma^a_{cb} = \left(\nabla_c \nabla_b \xi^a + K^a_{dcb}\xi^d\right)\varepsilon - \left[\nabla_c(h_{bex}\xi^x) + \nabla_b(h_{cex}\xi^x) - \nabla_e(h_{cbx}\xi^x)\right]g^{ea}\varepsilon, \tag{3.11}$$

Because of (1.8), A submanifolds variation is classified as affine when $\delta\Gamma^a_{cb} = 0$.

Now, we have the following:

Theorem 3.1 A complex isometric variation of a compact invariant submanifold M^n within a Kaehlerian manifold requires a fundamental holomorphic structure.

Proof. Considering equations (1.14), (3.3) and (3.6), we obtain the subsequent inter connections:

$$T_{cb} + f^e_c f^d_b T_{ed} = 0, \tag{3.12}$$

$$T_{cb} + T_{bc} = 0 \tag{3.13}$$

And hence

$$h^{cb}_x \xi^x T_{cb} = 0. \tag{3.14}$$

We now calculate $T_{cb}T^{cb}$, then by using (3.13), we get:

$$T_{cb}T^{cb} = \frac{1}{2}T^{cb}\left(T_{cb} - T_{bc}\right) \tag{3.15}$$

Again, by using (3.3), (3.14) and (3.12), we get:

$$T_{cb}T^{cb} = \frac{1}{2}T^{cb}\left[\nabla_c \xi_b - \nabla_b \xi_c - f^e_c f^d_b\left(\nabla_e \xi_d - \nabla_d \xi_e\right)\right] = 2T^{cb}\nabla_c \xi_b.$$

Conversely, when we operate on equation (3.3) with the operator ∇^c and make use of the condition $\nabla_c f^a_h = 0$, we find:

$$\nabla^c T_{cb} = \nabla^c \nabla_c \xi_b - \frac{1}{2}f^{ce}f^d_b\left(\nabla_c \nabla_e \xi_d - \nabla_e \nabla_c \xi_d\right) - 2\nabla^c(h_{cbx}\xi^x),$$

From which, using the Ricci-identity,

$\nabla^c T_{cb} = \nabla^c \nabla_c \xi_b + \frac{1}{2} f^{ce} f_b^d K_{ced}^a \xi_a - 2\nabla^c (h_{cbx} \xi^x),$

Also, by using (1.17), we get:

$$\nabla^c T_{cb} = \nabla^c \nabla_c \xi_b + K_b^a \xi_a - 2\nabla^c (h_{cbx} \xi^x). \tag{3.16}$$

Confirming that an isometric variation is indeed affine, we consequently establish that:

$\nabla_c \nabla_b \xi^a + K_{dcb}^a \xi^d - \left[\nabla_c (h_{bx}^a \xi^x) + \nabla_b (h_{cx}^a \xi^x) - \nabla^a (h_{cbx} \eta \xi^x)\right] = 0$

Because of (3.11), from which:

$\nabla^c \nabla_c \xi^a + K_d^a \xi^d - 2\nabla^c (h_{cx}^a \xi^x) = 0$

Because of (1.13). Therefore, $\nabla^c T_{cb} = 0$. From this fact and (3.15), we get:

$\nabla^c \left(\mathrm{T}_{cb} \xi^b\right) = \frac{1}{2} T_{cb} T^{cb}$.

By integrating this expression over the manifold M^n, we observe that $T_{cb} = 0$, leading to the conclusion that the variation is holomorphic, as indicated in (ii). This concludes the proof.

Reference

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